Supersymmetric Quantum Mechanics, multiphoton algebras and coherent states

Juan D García-Muñoz[1,2*], **David J Fernández C**[1] **and F Vergara-Méndez**[1,3]

[1]Physics Department, Cinvestav, P.O.B. 14-740, 07000 Mexico City, Mexico
[2]Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C-3, Ciudad Universitaria, Francisco J. Mújica S/N Col. Felícitas del Río, 58040 Morelia, Michoacán, México.
[3]Instituto Thomas Jefferson Campus Santa Mónica, Gardenia 5, Ex-hacienda de Santa Monica, 54050 Tlanepantla de Baz, Estado de México, México.

Email: juan.domingo.garcia@umich.mx, david.fernandez@cinvestav.mx and fer335@gmail.com

Abstract

The multiphoton algebras for one-dimensional Hamiltonians with infinite discrete spectrum, and for their associated kth-order SUSY partners are studied. In both cases, such an algebra is generated by the multiphoton annihilation and creation operators, as well as by Hamiltonians which are functions of an appropriate number operator. The algebras obtained turn out to be polynomial deformations of the corresponding single-photon algebra previously studied. The Barut-Girardello coherent states, which are eigenstates of the annihilation operator, are obtained and their uncertainty relations are explored by means of the associated quadratures.

Keywords: Supersymmetric Quantum Mechanics, multiphoton algebras, coherent states

I Introduction

At the end of the 20th century Mielnik introduced a generalization of the factorization method which represented a breakthrough on the subject. In fact, he showed that the standard annihilation and creation operators factorizing the harmonic oscillator Hamiltonian in a given order are not unique [1]. Moreover, if the initial product of the generalized factorization operators is permuted, then a new Hamiltonian appears, which has the same spectrum as the harmonic oscillator but different eigenstates. Later on, another authors generalized the idea raised by Mielnik in several ways, thus generating new solvable Hamiltonians through the factorization method. This led to identify an equivalent method called supersymmetric quantum mechanics (SUSY QM), by means of which the SUSY partner Hamiltonians are generated. These SUSY partner Hamiltonians have spectra slightly different from the initial one, but sharing an isoespectral part. Among the vast literature that exists about SUSY QM we recommend [2–13].

On the other hand, the SUSY partner Hamiltonians of the harmonic oscillator besides the corresponding annihilation and creation operators, generate what nowadays are called polynomial Heisenberg algebras [14, 15]. It has been shown as well that the harmonic oscillator Hamiltonian, together with annihilation and creation operators that are the mth powers of a and $a^\dagger$ (usually called multiphoton operators) also generate a polynomial Heisenberg algebra [16, 17]. In addition,

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if for the SUSY partners of the harmonic oscillator Hamiltonian the multiphoton annihilation and creation operators are employed, then additional polynomial Heisenberg algebras can be generated. Let us note that the harmonic oscillator Hamiltonian and its SUSY partners are almost the only systems for which the corresponding algebras have been explored [14, 15, 18–21]. Thus, it is natural to ask ourselves if it is possible to implement a similar approach for other one dimensional potentials (see e.g. [22]). Another natural issue is to figure out which algebras rule these systems, and their corresponding SUSY partners, if multiphoton annihilation and creation operators are used.

In this paper we will try to answer these questions. In order to do that, the article has been organized as follows. In section II it is sketched how to build the kth-order SUSY partners H_k departing from the initial Hamiltonian H_0, which has an infinite discrete spectrum. Then, in section III we will explore the multiphoton algebra associated to these Hamiltonians together with the corresponding multiphoton annihilation and creation operators. In section IV we will generate the Barut-Girardello coherent states for this algebra, and we will explore the uncertainty relation for the natural quadratures. Then, in section V we will focus on the harmonic oscillator and trigonometric Pöschl-Teller potentials as particular cases. Finally, in section VI we close the paper with our conclusions.

II kth-order supersymmetric quantum mechanics

Let us consider a chain of one-dimensional Hamiltonians of the form

$$H_j = -\frac{1}{2}\frac{d^2}{dx^2} + V_j(x), \quad j = 0, 1, \ldots, k,$$

(1)

with k being a natural number. It is said that the Hamiltonian H_k is a kth-order supersymmetric partner of the initial one H_0 if both are intertwined as follows

$$H_k B_k^\dagger = B_k^\dagger H_0,$$

(2)

where $B_k^\dagger$ is a kth-order differential intertwining operator which is factorized in the way

$$B_k^\dagger = b_k^\dagger \cdots b_1^\dagger,$$

(3)

with $b_j^\dagger$ being the first-order differential operator

$$b_j^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \beta_j\left(x, \epsilon_j\right)\right),$$

(4)

whose adjoint operator is

$$b_j = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \beta_j\left(x, \epsilon_j\right)\right),$$

(5)

where the functions $\beta_j(x, \epsilon_j)$ are real. The quantities ϵ_j are called factorization energies, which for simplicity will be taken as $\epsilon_k < \cdots < \epsilon_2 < \epsilon_1 < E_0$, with E_0 denoting the ground state energy of H_0 [9]. Each factorization energy ϵ_j is associated to one operator $b_j^\dagger$, which satisfies an intertwining relation similar to eq. (2) but involving H_j and H_{j-1}, i.e., the Hamiltonian H_k is the result of an iterative process leading to

$$H_0 = b_1 b_1^\dagger + \epsilon_1, \ldots, H_j = b_j^\dagger b_j + \epsilon_j = b_{j+1} b_{j+1}^\dagger + \epsilon_{j+1}, \ldots, H_k = b_k^\dagger b_k + \epsilon_k.$$

(6)

Let us suppose that the Hamiltonian H_0 is solvable, i.e., we know its eigenvalues E_n and eigenfunctions $\psi_n^{(0)}$. From the intertwining relation (2) we realize that the Hamiltonian H_k has also E_n as its eigenvalue, while the corresponding eigenfunctions $\psi_n^{(k)}$ and $\psi_n^{(0)}$ are related as follows

$$\psi_n^{(k)} = \frac{B_k^\dagger \psi_n^{(0)}}{\sqrt{(E_n - \epsilon_1)\dots(E_n - \epsilon_k)}}, \quad \psi_n^{(0)} = \frac{B_k \psi_n^{(k)}}{\sqrt{(E_n - \epsilon_1)\dots(E_n - \epsilon_k)}}. \tag{7}$$

Furthermore, the products of the operator $B_k^\dagger$ and its adjoint B_k turn out to be

$$B_k^\dagger B_k = (H_k - \epsilon_1)\dots(H_k - \epsilon_k), \quad B_k B_k^\dagger = (H_0 - \epsilon_1)\dots(H_0 - \epsilon_k). \tag{8}$$

It is worth noticing that there exist k functions $\psi_{\epsilon_j}^{(k)}$ that are in the *kernel* of the operators B_k and $B_k^\dagger B_k$, i.e., $\psi_{\epsilon_j}^{(k)}$ is a solution of the eigenvalue equation for the Hamiltonian H_k associated to the factorization energy ϵ_j. If such solutions are square-integrable the factorization energies ϵ_j must lie in the spectrum of the kth-order SUSY partner Hamiltonian H_k, thus $\mathrm{Sp}(H_k) = \{\epsilon_k, \dots, \epsilon_1, E_n, n = 0, 1, 2, \dots\}$. In figure 1 it is sketched the energy levels of both Hamiltonians H_0 and H_k as well as the intertwinings linking them.

On the other hand, the functions $\beta_j(x, \epsilon_j)$ fulfil the finite difference formula

$$\beta_j(x, \epsilon_j) = -\beta_{j-1}(x, \epsilon_{j-1}) - \frac{2(\epsilon_{j-1} - \epsilon_j)}{\beta_{j-1}(x, \epsilon_{j-1}) - \beta_{j-1}(x, \epsilon_j)}, \quad j = 2, 3, \dots, k. \tag{9}$$

Thus, and the end we only need k functions $\beta_1(x, \epsilon_j)$ which are solutions of the initial Riccati equation

$$\beta_1'(x, \epsilon_j) + \beta_1^2(x, \epsilon_j) = 2(V_0 - \epsilon_j), \tag{10}$$

where $f'(x) \equiv df(x)/dx$. If we make the change of variable $\beta_1(x, \epsilon_j) = u_j'/u_j$, the previous equation can be written in the most familiar form
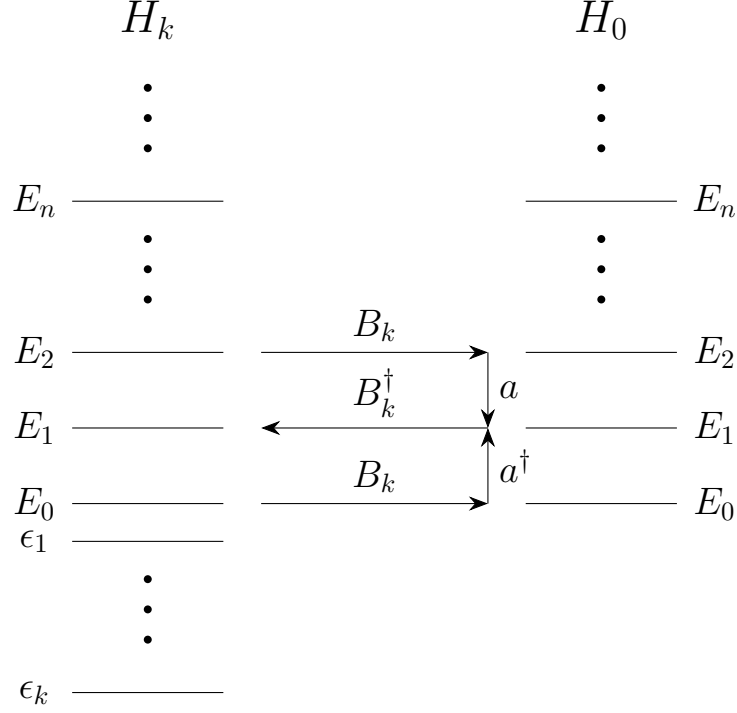
$$-\frac{1}{2}u_j'' + V_0(x)u_j = \epsilon_j u_j, \tag{11}$$

i.e., $u_j \equiv u(x, \epsilon_j)$ is a solution of the stationary Schrödinger equation for the initial Hamiltonian H_0. Meanwhile, the potential $V_k(x)$ is given by

$$V_k(x) = V_0 - \sum_{j=1}^{k} \beta_j'(x, \epsilon_j) = V_0 - \{\log[W(u_1, \dots, u_k)]\}''. \tag{12}$$

III Multiphoton algebras of H_0 and H_k

In order to address the multiphoton algebras of H_0 and H_k it is important to study first the intrinsic algebraic structure of H_0.

Figure 1: Spectral representation of H_0, H_k and the intertwinings linking them.

1 Intrinsic algebra of H_0

We are going to consider one-dimensional Hamiltonians with the form given in eq. (1) for $j = 0$. We suppose that H_0 is solvable, i.e., its eigenvalues E_n and eigenstates $|\psi_n\rangle$ are known. Moreover, the eigenvalues E_n are ordered in the standard way, and define an explicit dependence on the principal quantum number n, i.e., $E_n \equiv E(n)$.

The intrinsic annihilation and creation operators a and $a^\dagger$ are defined through their action onto the states $|\psi_n\rangle$ as follows

$$a|\psi_n\rangle = r(n)|\psi_{n-1}\rangle, \ a^\dagger|\psi_n\rangle = \overline{r}(n+1)|\psi_{n+1}\rangle, \ r(n) = e^{i\alpha(E_n - E_{n-1})}\sqrt{E_n - E_0}, \ \alpha \in \mathbb{R}, \quad (13)$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Thus, the set of operators $\{H_0, a, a^\dagger\}$ generates what is called intrinsic algebra for the system, which is defined through the following commutators

$$[a, a^\dagger] = f(N), \ [H_0, a^\dagger] = f(N-1)a^\dagger, \ [H_0, a] = -f(N)a, \ f(N) = E(N+1) - E(N), \quad (14)$$

with $f(N)$ being a function of the intrinsic number operator N which fulfils

$$N|\psi_n\rangle = n|\psi_n\rangle, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (15)$$

2 Multiphoton algebra of H_0

Let us consider now the multiphoton annihilation and creation operators $a_m, a_m^\dagger$ defined as the mth powers of the intrinsic ones

$$a_m = a^m, \quad a_m^\dagger = (a^\dagger)^m, \quad m \in \mathbb{N}, \quad (16)$$

whose actions onto the eigenstates $|\psi_n\rangle$ are given by

$$a_m|\psi_n\rangle = e^{i\alpha(E_n - E_{n-m})}\sqrt{(E_n - E_0)\ldots(E_{n-m+1} - E_0)}|\psi_{n-m}\rangle,$$
$$a_m^\dagger|\psi_n\rangle = e^{-i\alpha(E_{n+m} - E_n)}\sqrt{(E_{n+1} - E_0)\ldots(E_{n+m} - E_0)}|\psi_{n+m}\rangle. \tag{17}$$

Analogously to the intrinsic case, the multiphoton algebra is defined by the commutation rules

$$[a_m, a_m^\dagger] = \prod_{l=0}^{m-1}[E(N + m - l) - E_0] - \prod_{l=0}^{m-1}[E(N - l) - E_0], \tag{18}$$
$$[H_0, a_m^\dagger] = f_m(N - m)a_m^\dagger, \quad [H_0, a_m] = -f_m(N)a_m, \quad f_m(N) = E(N + m) - E(N).$$

It is important to define now the extremal states, which are eigenstates of the Hamiltonian H_0 that are also annihilated by the multiphoton annihilation operator, i.e., $a_m|\psi_n\rangle = 0$. The intrinsic algebra of H_0 is the particular case of the multiphoton algebra with $m = 1$, for which there exists only one extremal state, namely, the ground state $|\psi_0\rangle$. The iterated action of the intrinsic creation operator on such extremal state leads to the infinite ladder of eigenstates $|\psi_n\rangle$ of H_0. On the other hand, in order to identify the extremal states $|\psi_j\rangle$ for general $m > 1$, let us note that they are also in the *kernel* of the operator $a_m^\dagger a_m$, i.e.,

$$a_m^\dagger a_m|\psi_j\rangle = \prod_{l=0}^{m-1}[E(j - l) - E_0]|\psi_j\rangle = (E_j - E_0)(E_{j-1} - E_0)\cdots(E_{j-m+1} - E_0)|\psi_j\rangle = 0. \tag{19}$$
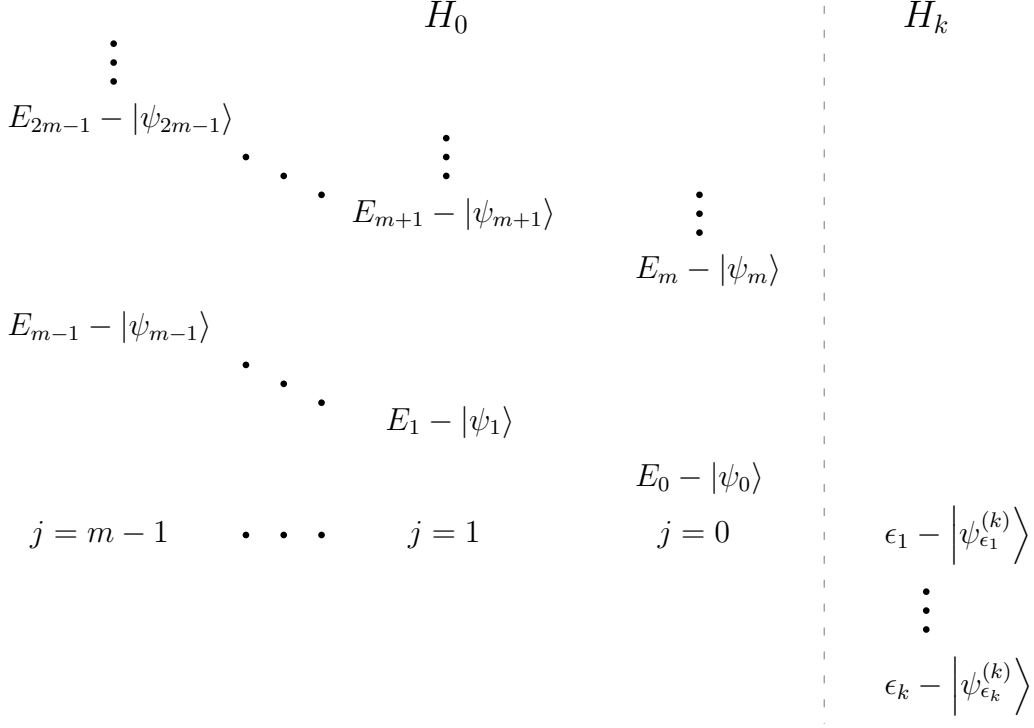
We can see that the eigenstates $|\psi_j\rangle$ of H_0 with $j = 0, 1, ..., m - 1$ are as well annihilated by a_m, thus the extremal states are the first m energy eigenstates of H_0. Moreover, the iterated action of the multiphoton creation operator on each extremal state $|\psi_j\rangle$ produces subsequent states, denoted as $|\psi_{j+nm}\rangle$, whose energies E_{j+nm} at the end arranges as j infinite ladders. We note that such energy ladders do not only split the spectrum of the system, but also divide the state space in m orthogonal subspaces, such that their direct sum is the original space. In figure 2 it is represented the spectrum of H_0, divided in m infinite independent ladders labeled by the index j. Let us note as well that the energy levels of H_0 do not have to be equidistant, although in such figure we are showing them in this way for illustration purposes.

3 Multiphoton algebra of H_k

For the kth SUSY partner Hamiltonian H_k the natural multiphoton annihilation and creation operators are given by

$$\ell_{k,m} = B_k^\dagger a_m B_k, \quad \ell_{k,m}^\dagger = B_k^\dagger a_m^\dagger B_k, \tag{20}$$

where $B_k, B_k^\dagger$ are the intertwining operators of the previous section while $a_m, a_m^\dagger$ are the multiphoton annihilation and creation operators for the initial Hamiltonian H_0. We can find the action of

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$$H_0 \qquad\qquad H_k$$

$$\vdots$$
$$E_{2m-1} - |\psi_{2m-1}\rangle$$
$$\vdots$$
$$E_{m+1} - |\psi_{m+1}\rangle$$
$$E_m - |\psi_m\rangle$$
$$E_{m-1} - |\psi_{m-1}\rangle$$
$$\vdots$$
$$E_1 - |\psi_1\rangle$$
$$E_0 - |\psi_0\rangle$$
$$j = m-1 \qquad \cdots \qquad j = 1 \qquad j = 0 \qquad\qquad \epsilon_1 - \left|\psi_{\epsilon_1}^{(k)}\right\rangle$$
$$\vdots$$
$$\epsilon_k - \left|\psi_{\epsilon_k}^{(k)}\right\rangle$$

Figure 2: Spectrum of H_0, divided into m infinite ladders labeled by the index j. The spectrum of H_k has a similar splitting as $\mathrm{Sp}(H_0)$, but it contains k additional single-step ladders corresponding to the factorization energies ϵ_i.

$\ell_{k,m}$, $\ell_{k,m}^\dagger$ onto the states $\left|\psi_n^{(k)}\right\rangle$ by using eqs. (7, 17), leading to

$$
\ell_{k,m}\left|\psi_n^{(k)}\right\rangle = e^{i\alpha(E_n - E_{n-m})}\sqrt{\prod_{l=0}^{m-1}(E_{n-l} - E_0)\prod_{i=1}^{k}(E_n - \epsilon_i)(E_{n-m} - \epsilon_i)}\left|\psi_{n-m}^{(k)}\right\rangle,
$$

$$
\ell_{k,m}^\dagger\left|\psi_n^{(k)}\right\rangle = e^{-i\alpha(E_{n+m} - E_n)}\sqrt{\prod_{l=0}^{m-1}(E_{n+m-l} - E_0)\prod_{i=1}^{k}(E_n - \epsilon_i)(E_{n+m} - \epsilon_i)}\left|\psi_{n+m}^{(k)}\right\rangle. \tag{21}
$$

Before exploring the multiphoton algebra generated by the operator set $\{H_k, \ell_{k,m}, \ell_{k,m}^\dagger\}$, let us identify the number operator N_k, such that its action onto the states $\left|\psi_n^{(k)}\right\rangle$ is

$$
N_k\left|\psi_n^{(k)}\right\rangle = n\left|\psi_n^{(k)}\right\rangle. \tag{22}
$$

This operator can be written in terms of the intrinsic number operator N of the initial system defined in equation (15), whose explicit expression becomes

$$
N_k = B_k^\dagger \frac{N}{(E(N) - \epsilon_1)\dots(E(N) - \epsilon_k)} B_k. \tag{23}
$$

Since the states $\left|\psi_{\epsilon_i}^{(k)}\right\rangle$ are in the *kernel* of B_k, then $N_k\left|\psi_{\epsilon_i}^{(k)}\right\rangle = 0$, due to they are not characterized by the quantum number n but by the factorization energy ϵ_i.

By using eq. (21), the multiphoton algebra generated by $\{H_k, \ell_{k,m}, \ell_{k,m}^\dagger\}$ is defined by the commutation rules

$$[\ell_{k,m}, \ell_{k,m}^\dagger] = \left\{ \prod_{l=0}^{m-1}[E(N_k + m - l) - E_0] \prod_{i=1}^{k}[E(N_k + m) - \epsilon_i][E(N_k) - \epsilon_i] - \right.$$
$$\left. - \prod_{l=0}^{m-1}[E(N_k - l) - E_0] \prod_{i=1}^{k}[E(N_k - m) - \epsilon_i][E(N_k) - \epsilon_i] \right\} \mathcal{P}_{iso}^{(k)}, \quad (24)$$

$$[H_k, \ell_{k,m}^\dagger] = f_m(N_k - m)\ell_{k,m}^\dagger, \quad [H_k, \ell_{k,m}] = -f_m(N_k)\ell_{k,m}, \quad (25)$$

where $\mathcal{P}_{iso}^{(k)} = \sum_{n=0}^{\infty}|\psi_n^{(k)}\rangle\langle\psi_n^{(k)}|$ is the projector onto the subspace $\mathcal{H}_{iso}^{(k)}$ generated by the eigenstates of H_k associated to the isospectral part of H_0. Let us recall that the extremal states belong to the *kernel* of the operator product $\ell_{k,m}^\dagger \ell_{k,m}$, which is given by

$$\ell_{k,m}^\dagger \ell_{k,m} = \prod_{l=0}^{m-1}[E(N_k - l) - E_0]\prod_{i=1}^{k}[E(N_k - m) - \epsilon_i][E(N_k) - \epsilon_i]\mathcal{P}_{iso}^{(k)}. \quad (26)$$

Since they are as well eigenstates of H_k, then there are $k + m$ extremal states. The first m of them are $\left|\psi_j^{(k)}\right\rangle$, $j = 0, 1, \cdots, m-1$, whose energies E_j are in the isospectral part of H_0 and H_k. The iterated action of $\ell_{k,m}^\dagger$ on each $|\psi_j^{(k)}\rangle$ generates all the states $\left|\psi_{j+nm}^{(k)}\right\rangle$ of the associated ladder, with energies E_{j+nm}. The other k extremal states become $\left|\psi_{\epsilon_i}^{(k)}\right\rangle$, $i = 1, 2, \cdots, k$, where each energy ϵ_i constitutes by itself a single-step ladder, due to $\ell_{k,m}^\dagger\left|\psi_{\epsilon_i}^{(k)}\right\rangle = 0$. In figure 2 it is represented the spectrum of H_k, which is divided into m infinite ladders labeled by the index j, and the k single-step ladders.

IV Coherent states

The coherent states for the intrinsic algebra of H_0 and the natural algebra of H_k have been generated previously [22]. Let us derive now the coherent states for the multiphoton algebra of H_0 and H_k, which will be called multiphoton coherent states (MCS).

1 MCS of H$_0$

Let us look for the MCS $|z\rangle$ of H_0 as eigenstates of the multiphoton annihilation operator of eq. (16). Since the state space is divided into m orthogonal subspaces, such that their direct sum is the original space, then the coherent states $|z\rangle$ in general will be linear combinations of m orthogonal coherent states (CS) $|z\rangle_j$, where each $|z\rangle_j$ belongs just to the j-th subspace. Thus, let us restrict ourselves to look for the coherent states $|z\rangle_j$ as eigenstates of the multiphoton annihilation operator [23], i.e.,

$$a_m|z\rangle_j = z|z\rangle_j, \quad z \in \mathbb{C}. \quad (27)$$

Since $|z\rangle_j$ is a lineal combination of the states $|\psi_{j+nm}\rangle$, a straightforward calculation leads to

$$|z\rangle_j = \frac{1}{\sqrt{\sum\limits_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_n^j}}} \sum_{n=0}^{\infty} e^{-i\alpha(E_{j+nm}-E_j)} \frac{z^n}{\sqrt{\rho_n^j}} |\psi_{j+nm}\rangle, \tag{28}$$

where

$$\rho_n^j = \begin{cases} \prod\limits_{l=0}^{nm-1} (E_{j+nm-l} - E_0) & \text{for} \quad n > 0, \\ 1 & \text{for} \quad n = 0. \end{cases} \tag{29}$$

It is worth noticing that the coherent state $|z\rangle_j$ reduces to the extremal state $|\psi_j\rangle$ for $z = 0$.

A characteristic feature of the standard harmonic oscillator coherent states is that they minimize the Heisenberg uncertainty relation for the standard position and momentum operators. However, in general it is difficult to calculate algebraically in a closed form the action of such operators on the CS of eq. (28). Moreover, trying to express the standard position and momentum operators in terms of the intrinsic or multiphoton annihilation and creation operators of eqs. (13) and (16) is a non-trivial task that goes beyond the scope of this paper. Thus, we shall calculate the uncertainty relation for the naturally associated quadratures, i.e., we will define two operators analogous to the standard position and momentum operators which will be expressed either in terms of a and $a^\dagger$ or a_m and $a_m^\dagger$. Consequently, we will be able to calculate algebraically such uncertainty product. Let us explore first the quadratures defined by

$$X = \frac{1}{\sqrt{2}}(a^\dagger + a), \quad P = \frac{i}{\sqrt{2}}(a^\dagger - a). \tag{30}$$

By calculating their corresponding uncertainty relation on the multiphoton coherent state $|z\rangle_j$, which in general is not an eigenstate of a, we get that

$$(\Delta X)(\Delta P) = \begin{cases} |a|z\rangle_j|^2 + \frac{1}{2}\langle[a,a^\dagger]\rangle_j & \text{for} \quad m \geq 3, \\ \sqrt{(|a|z\rangle_j|^2 + \frac{1}{2}\langle[a,a^\dagger]\rangle_j)^2 - (\text{Re}(z))^2} & \text{for} \quad m = 2, \\ \frac{1}{2}\langle[a,a^\dagger]\rangle_0 & \text{for} \quad m = 1, \end{cases} \tag{31}$$

with

$$|a|z\rangle_j|^2 = \langle a^\dagger a \rangle_j = \frac{1}{\sum\limits_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_n^j}} \sum_{n=0}^{\infty} \frac{E_{j+nm} - E_0}{\rho_n^j} |z|^{2n},$$

$$\langle[a,a^\dagger]\rangle_j = \frac{1}{\sum\limits_{n=0}^{\infty} \frac{|z|^{2n}}{\rho_n^j}} \sum_{n=0}^{\infty} \frac{E_{j+nm+1} - E_{j+nm}}{\rho_n^j} |z|^{2n}. \tag{32}$$

The other case to be studied arises when defining the operators

$$X_m = \frac{1}{\sqrt{2}}(a_m^\dagger + a_m), \quad P_m = \frac{i}{\sqrt{2}}(a_m^\dagger - a_m). \tag{33}$$

Since the multiphoton coherent states $|z\rangle_j$ are eigenstates of a_m it turns out that

$$\left(\Delta X_m\right)\left(\Delta P_m\right) = \frac{\langle a_m a_m^\dagger\rangle_j - |z|^2}{2},$$ (34)

where

$$\langle a_m a_m^\dagger\rangle_j = \frac{1}{\sum\limits_{n=0}^\infty \frac{|z|^{2n}}{\rho_n^j}} \sum_{n=0}^\infty \frac{|z|^{2n}}{\rho_n^j} \prod_{l=0}^{m-1}(E_{j+(n+1)m-l} - E_0).$$ (35)

2 MCS of $\mathrm{H_k}$

In a similar way, let us look for the MCS $|w\rangle_j$ as eigenstates of the multiphoton annihilation operator $\ell_{k,m}$ of eq. (20), namely,

$$\ell_{k,m}|w\rangle_j = w|w\rangle_j, \quad w \in \mathbb{C}.$$ (36)

They are lineal combinations of the states $\left|\psi_n^{(k)}\right\rangle$ but the isolated states $\left|\psi_{\epsilon_i}^{(k)}\right\rangle$ do not contribute to these superpositions since they are in the *kernel* of the operator B_k. Thus, it turns out that

$$|w\rangle_j = \frac{1}{\sqrt{\sum\limits_{n=0}^\infty \frac{|w|^{2n}}{\varphi_n^j}}} \sum_{n=0}^\infty e^{-i\alpha(E_{j+nm}-E_j)}\frac{w^n}{\sqrt{\varphi_n^j}}\left|\psi_{j+nm}^{(k)}\right\rangle,$$ (37)

with φ_n^j being given by

$$\varphi_n^j = \begin{cases} \dfrac{\prod_{i=1}^k \prod_{p=0}^n (E_{j+pm}-\epsilon_i)^2 \prod_{l=0}^{nm-1}(E_{j+nm-l}-E_0)}{\prod_{i=1}^k (E_{j+nm}-\epsilon_i)(E_j-\epsilon_i)} & \text{for} \quad n > 0, \\ 1 & \text{for} \quad n = 0. \end{cases}$$ (38)

Once again, for $w = 0$ the MCS become the extremal states $\left|\psi_j^{(k)}\right\rangle$.

The quadratures approach allows us to calculate algebraically the uncertainty relation for the operators X_k and P_k defined in terms of $\ell_{k,1} = \ell_k$ and $\ell_{k,1}^\dagger = \ell_k^\dagger$ as follows

$$X_k = \frac{1}{\sqrt{2}}(\ell_k^\dagger + \ell_k), \quad P_k = \frac{i}{\sqrt{2}}(\ell_k^\dagger - \ell_k).$$ (39)

Thus, for the MCS $|w\rangle_j$ which are not necessarily eigenstates of ℓ_k, the uncertainty relation becomes analogous to the one for the quadratures X and P, but now it is simpler to express the result in terms of the products $\ell_k^\dagger \ell_k$ and $\ell_k \ell_k^\dagger$ instead of the commutator $[\ell_k, \ell_k^\dagger]$, namely,

$$(\Delta X_k)(\Delta P_k) = \begin{cases} \frac{\langle \ell_k^\dagger \ell_k\rangle_j + \langle \ell_k \ell_k^\dagger\rangle_j}{2} & \text{for} \quad m \geq 3, \\ \sqrt{\frac{(\langle \ell_k^\dagger \ell_k\rangle_j + \langle \ell_k \ell_k^\dagger\rangle_j)^2}{4} - (\mathrm{Re}(w))^2} & \text{for} \quad m = 2, \\ \frac{\langle \ell_k \ell_k^\dagger\rangle_j - |w|^2}{2} & \text{for} \quad m = 1, \end{cases}$$ (40)

where

$$\langle \ell_k^\dagger \ell_k \rangle_j = |\ell_k |w\rangle_j|^2 = \frac{\sum\limits_{n=0}^{\infty} \frac{(E_{j+nm}-E_0)\prod\limits_{i=1}^{k}(E_{j+nm}-\epsilon_i)(E_{j+nm-1}-\epsilon_i)}{\varphi_n^j}|w|^{2n}}{\sum\limits_{n=0}^{\infty} \frac{|w|^{2n}}{\varphi_n^j}},$$

$$\langle \ell_k \ell_k^\dagger \rangle_j = |\ell_k^\dagger |w\rangle_j|^2 = \frac{\sum\limits_{n=0}^{\infty} \frac{(E_{j+nm+1}-E_0)\prod\limits_{i=1}^{k}(E_{j+nm+1}-\epsilon_i)(E_{j+nm}-\epsilon_i)}{\varphi_n^j}|w|^{2n}}{\sum\limits_{n=0}^{\infty} \frac{|w|^{2n}}{\varphi_n^j}}.$$

(41)

Finally, let us define the quadratures $X_{k,m}$ and $P_{k,m}$ in terms of the multiphoton annihilation and creation operators as follows

$$X_{k,m} = \frac{1}{\sqrt{2}}(\ell_{k,m}^\dagger + \ell_{k,m}), \quad P_{k,m} = \frac{i}{\sqrt{2}}(\ell_{k,m}^\dagger - \ell_{k,m}).$$

(42)

Thus, the associated uncertainty relation on the MCS $|w\rangle_j$, which is an eigenstate of $\ell_{k,m}$, becomes

$$(\Delta X_{k,m})(\Delta P_{k,m}) = \frac{\langle \ell_{k,m} \ell_{k,m}^\dagger \rangle_j - |w|^2}{2},$$

(43)

with

$$\langle \ell_{k,m} \ell_{k,m}^\dagger \rangle_j = |\ell_{k,m}^\dagger |w\rangle_j|^2 = \frac{\sum\limits_{n=0}^{\infty} \frac{\prod\limits_{l=0}^{m-1}(E_{j+(n+1)m-l}-E_0)\prod\limits_{i=1}^{k}(E_{j+(n+1)m}-\epsilon_i)(E_{j+nm}-\epsilon_i)}{\varphi_n^j}|w|^{2n}}{\sum\limits_{n=0}^{\infty} \frac{|w|^{2n}}{\varphi_n^j}}.$$

(44)

Since from the very beginning we assumed that the spectrum of the initial Hamiltonian is increasing with respect to n, such that $E_{n+1} > E_n \ \forall \ n$, and for the SUSY partner potentials the factorization energies are below E_0, then all infinite series resulting for the uncertainty relations converge. This is true, in particular, for the examples that we will work in the next section, where the energy eigenvalues of H_0 will be linear or quadratic functions of n, i.e., $E(n) \sim n, n^2$.

V Particular cases

The theory of the previous sections allows us to determine the multiphoton algebra and MCS for most of the known solvable potentials and their kth order SUSY partners. In this work we will address two examples: the harmonic oscillator and trigonometric Pöschl-Teller potentials. We shall characterize their corresponding algebras and we will illustrate the uncertainty relations for the quadratures associated to the MCS.

1 Harmonic oscillator

The harmonic oscillator potential is given by $V_0(x) = x^2/2$, whose Hamiltonian is linear in N, $H_0 = E(N) = N + 1/2$. When substituting this in eq. (18) the multiphoton algebra looks like

$$[a_m, a_m^\dagger] = \prod_{l=0}^{m-1}\left(H_0 + m - l - \frac{1}{2}\right) - \prod_{l=0}^{m-1}\left(H_0 - l - \frac{1}{2}\right),$$

$$[H_0, a_m^\dagger] = m a_m^\dagger, \quad [H_0, a_m] = -m a_m. \tag{45}$$

Meanwhile, the algebra for its SUSY partners becomes

$$[\ell_{k,m}, \ell_{k,m}^\dagger] = \left\{ \prod_{l=0}^{m-1}\left(H_k + m - l - \frac{1}{2}\right) \prod_{i=1}^{k}\left(H_k + m - \epsilon_i\right)\left(H_k - \epsilon_i\right) \right.$$

$$\left. - \prod_{l=0}^{m-1}\left(H_k - l - \frac{1}{2}\right) \prod_{i=1}^{k}\left(H_k - m - \epsilon_i\right)\left(H_k - \epsilon_i\right) \right\} \mathcal{P}_{iso}^{(k)}, \tag{46}$$

$$[H_k, \ell_{k,m}^\dagger] = m \ell_{k,m}^\dagger, \quad [H_k, \ell_{k,m}] = -m \ell_{k,m}. \tag{47}$$

Note that in both algebras the commutator between the annihilation and creation operators is a polynomial in the corresponding Hamiltonian in the subspace associated to the initial spectrum, which for H_0 coincides with the full Hilbert space. Furthermore, the commutator between the Hamiltonian and a ladder operator is proportional to the same ladder operator, the factor being the difference between two consecutive energy levels in the same ladder which here is a constant (see figure 2).

On the other hand, the coherent states $|z\rangle_j$ for the oscillator potential $V_0(x)$ with $\alpha = 0$ and $m = 1$ $(j = 0)$ reduce to the standard ones

$$|z\rangle_0 = e^{-\frac{|z|^2}{2}} e^{z a^\dagger} |\psi_0\rangle, \tag{48}$$

while for $m = 2$ the so-called "even" and "odd" CS are recovered, which are given by

$$|z\rangle_0 = \sqrt{\mathrm{sech}|z|} \sum_{n=0}^{\infty} \frac{z^n |\psi_{2n}\rangle}{\sqrt{(2n)!}}, \quad |z\rangle_1 = \sqrt{|z|\mathrm{csch}|z|} \sum_{n=0}^{\infty} \frac{z^n |\psi_{2n+1}\rangle}{\sqrt{(2n+1)!}}. \tag{49}$$

For the SUSY partners with $k = 1$, $\epsilon = -1/2$ and $m = 2$, the "even" and "odd" CS are given by

$$|w\rangle_0 = \frac{\sum_{n=0}^{\infty} \sqrt{\frac{(2n+1)}{(2n)![(2n+1)!!]^2}} w^n |\psi_{2n}^{(1)}\rangle}{\sqrt{\sum_{n=0}^{\infty} \frac{(2n+1)|w|^{2n}}{(2n)![(2n+1)!!]^2}}}, \quad |w\rangle_1 = \frac{\sum_{n=0}^{\infty} \sqrt{\frac{(2n+2)}{(2n+1)![(2n+2)!!]^2}} w^n |\psi_{2n+1}^{(1)}\rangle}{\sqrt{\sum_{n=0}^{\infty} \frac{(2n+2)|w|^{2n}}{(2n+1)![(2n+2)!!]^2}}}. \tag{50}$$

The uncertainty relations for the quadratures arising in eqs. (31) and (34), associated to the "even" and "odd" CS of the harmonic oscillator potential given in eq. (49) are shown in figure 3. In figure 4 it is plotted the uncertainty relations shown in eqs. (40) and (43) for the CS given in eq. (50) associated to the first-order SUSY partner potentials of the harmonic oscillator with factorization energy $\epsilon = -1/2$.
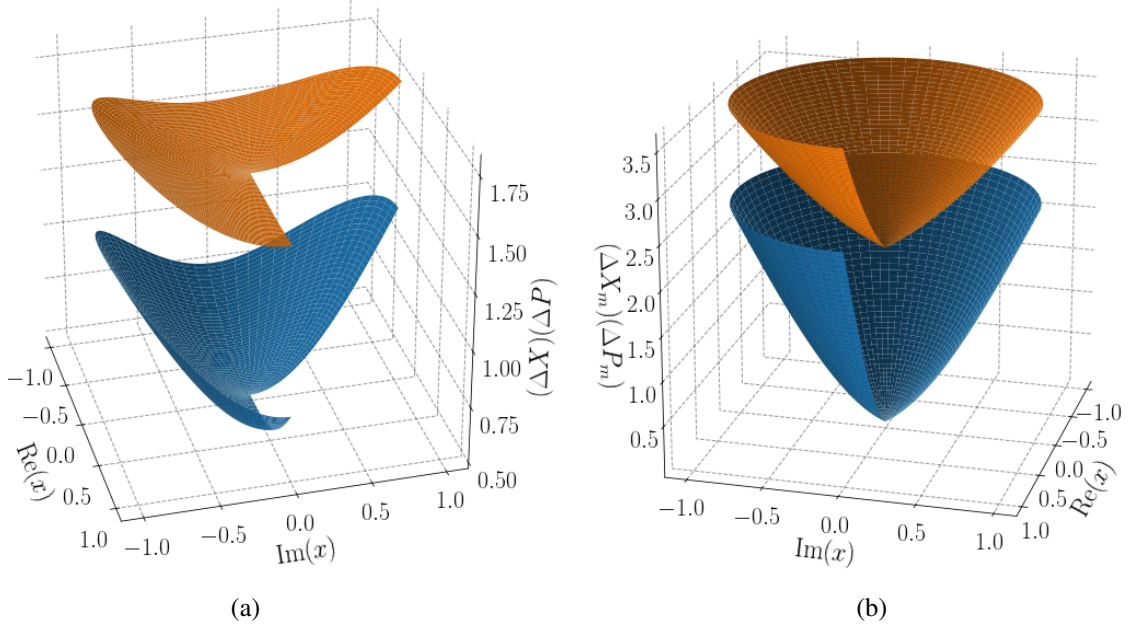
Figure 3: Uncertainty relations for the "even" and "odd" CS of the harmonic oscillator potential associated to the quadratures of the intrinsic annihilation and creation operators (a), where the minimum values are $1/2$ for the "even" state and $3/2$ for the "odd" one. In (b) the corresponding quantities for the multiphoton operators are shown, and the minimum values are now 0 for the "even" state and 2 for the "odd" one.

2 Trigonometric Pöschl-Teller potential

The initial potential is now $V_0(x) = \nu(\nu - 1)/(2\cos^2 x)$, $\nu > 1$. The energy operator in terms of the number operator becomes $H_0 = E(N) = (N + \nu)^2/2$, thus the commutators defining the multiphoton algebra of H_0 turn out to be

$$
\begin{aligned}
[a_m, a_m^\dagger] &= \prod_{l=0}^{m-1}\left(H_0 - (m-l)(N+l-m) - \tfrac{(m-l+\nu)^2}{2}\right) - \prod_{l=0}^{m-1}\left(H_0 - l(N-l) - \tfrac{(l+\nu)^2}{2}\right) \\
[H_0, a_m^\dagger] &= \frac{m}{2}[2(N+\nu) - m]a_m^\dagger, \quad [H_0, a_m] = -\frac{m}{2}[2(N+\nu) + m]a_m.
\end{aligned}
\tag{51}
$$

On the other hand, the algebra of the SUSY partner Hamiltonian H_k is characterized by

$$
\begin{aligned}
[\ell_{k,m}, \ell_{k,m}^\dagger] = \Bigg\{ &\prod_{l=0}^{m-1}\left[H_k - (m-l)(N_k + l - m) - \frac{(m-l+\nu)^2}{2}\right]\prod_{i=1}^{k}\left[H_k + \frac{m}{2}\right. \\
&\times (m + 2(N_k + \nu)) - \epsilon_i\Big](H_k - \epsilon_i) - \prod_{l=0}^{m-1}\left[H_k - l(N_k - l) - \tfrac{(l+\nu)^2}{2}\right] \\
&\times \prod_{i=1}^{k}\left[H_k + \tfrac{m}{2}(m - 2(N_k + \nu)) - \epsilon_i\right](H_k - \epsilon_i)\Bigg\}\mathcal{P}_{iso}^{(k)}, \\
[H_k, \ell_{k,m}^\dagger] = &\frac{m}{2}[2(N_k + \nu) - m]\ell_{k,m}^\dagger, \quad [H_k, \ell_{k,m}] = -\frac{m}{2}[2(N_k + \nu) + m]\ell_{k,m}.
\end{aligned}
\tag{52}
$$

$$
[H_k, \ell_{k,m}^\dagger] = \frac{m}{2}[2(N_k + \nu) - m]\ell_{k,m}^\dagger, \quad [H_k, \ell_{k,m}] = -\frac{m}{2}[2(N_k + \nu) + m]\ell_{k,m}.
\tag{53}
$$
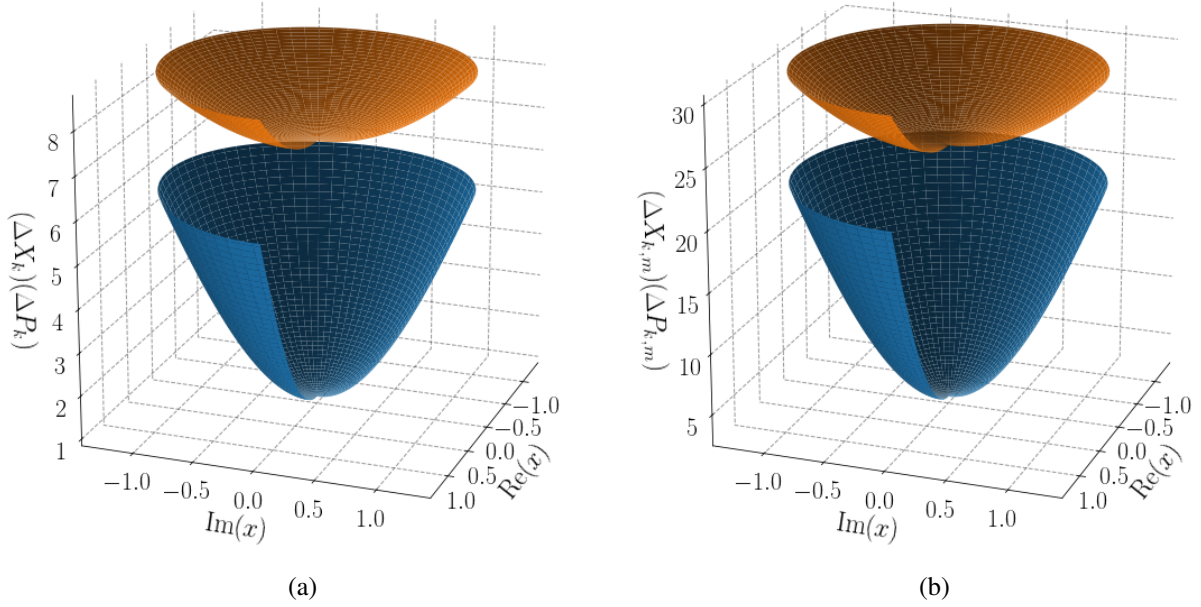
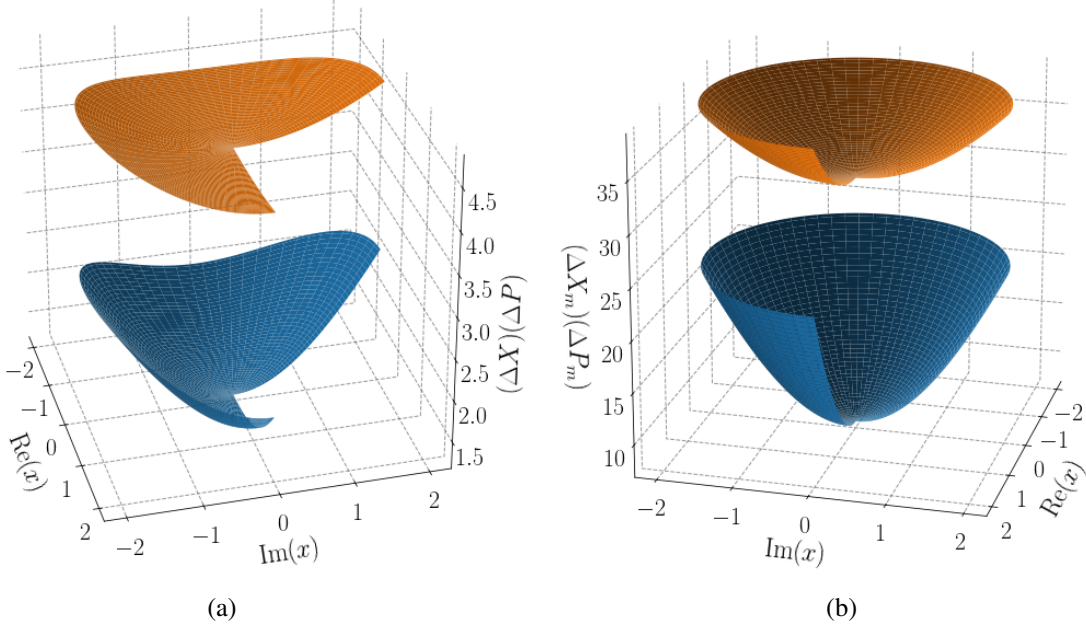**Figure 4:** Uncertainty relations for the "even" and "odd" CS of the first-order harmonic oscillator SUSY partner potential with factorization energy $\epsilon = -1/2$, for the quadratures associated to the annihilation and creation operators ℓ_k, $\ell_k^\dagger$ (a). The minimum values achieved are 1 for the "even" state and 7 for the "odd" one. In (b) the multiphoton operators $\ell_{k,m}$, $\ell_{k,m}^\dagger$ are used, and the minimum values obtained are 3 for the "even" state and 24 for the "odd" one.

Note that the algebras of eqs. (51, 52, 53) for the trigonometric Pöschl-Teller potential and its kth-order SUSY partners have non-constant factors in the commutators of the Hamiltonian with the corresponding ladder operators, thus defining non-constant energy gaps between two consecutive energy steps in the ladders (see figure 2). However, the commutators between the two ladder operators is a polynomial function in the corresponding Hamiltonians in the subspace associated to the initial spectrum.

The MCS for the trigonometric Pöschl-Teller potential $|z\rangle_j$ with $\alpha = 0$ and $m = 2$ are given by

$$|z\rangle_0 = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{2^n z^n |\psi_{2n}\rangle}{\sqrt{(2n)![2(n+\nu)]!}}}{\sqrt{\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}|z|^{2n}}{(2n)![2(n+\nu)]!}}}, \quad |z\rangle_1 = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{2^n z^n |\psi_{2n+1}\rangle}{\sqrt{(2n+1)![2(n+\nu)+1]!}}}{\sqrt{\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}|z|^{2n}}{(2n+1)![2(n+\nu)+1]!}}}. \tag{54}$$

Meanwhile, the corresponding CS $|w\rangle_j$ for the associated first-order SUSY partner with $k = 1$,

Figure 5: Uncertainty relations for the CS of the trigonometric Pöschl-Teller potential for the quadratures of the intrinsic annihilation and creation operators (a). The minimum values achieved are $5/4$ for $|z\rangle_0$ and $17/4$ for $|z\rangle_1$. In (b) the corresponding multiphoton operators are used, and the minimum values obtained are $15/2$ for $|z\rangle_0$ and $63/2$ for $|z\rangle_1$. The potential parameter taken is $\nu = 2$.

$\epsilon = -1/2$ and $m = 2$ become

$$|w\rangle_0 = \frac{\sum\limits_{n=0}^{\infty} \sqrt{\frac{2^{2n+1}[(2n+\nu)^2+1]}{(2n)![2(n+\nu)]! \prod\limits_{p=0}^{n}[(2p+\nu)^2+1]^2}}\, w^n |\psi_{2n}^{(1)}\rangle}{\sqrt{\sum\limits_{n=0}^{\infty}\frac{2^{2n+1}[(2n+\nu)^2+1]|w|^{2n}}{(2n)![2(n+\nu)]! \prod\limits_{p=0}^{n}[(2n+\nu)^2+1]^2}}},$$

$$|w\rangle_1 = \frac{\sum\limits_{n=0}^{\infty} \sqrt{\frac{2^{2n+1}[(2n+1+\nu)^2+1]}{(2n+1)![2(n+\nu)+1]! \prod\limits_{p=0}^{n}[(2p+1+\nu)^2+1]^2}}\, w^n |\psi_{2n+1}^{(1)}\rangle}{\sqrt{\sum\limits_{n=0}^{\infty}\frac{2^{2n+1}[(2n+1+\nu)^2+1]|w|^{2n}}{(2n+1)![2(n+\nu)+1]! \prod\limits_{p=0}^{n}[(2n+1+\nu)^2+1]^2}}}. \tag{55}$$

Figure 5 shows the uncertainty relations associated to the quadratures for the CS of the trigonometric Pöschl-Teller potential given in eq. (54). In figure 6, the uncertainty relations for the CS of the associated first-order SUSY partner given in eq. (55) are shown.

VI Conclusions

In this paper the multiphoton algebras ruling a class of one dimensional Hamiltonians with an infinite discrete spectrum, the corresponding kth-order SUSY partners and their associated multiphoton algebras, have been studied. In the examples addressed here these algebras turn out to be polynomial functions of the corresponding Hamiltonians in the subspace associated to the isospectral part. The energy gap between two adjacent energy levels is constant for the harmonic oscillator
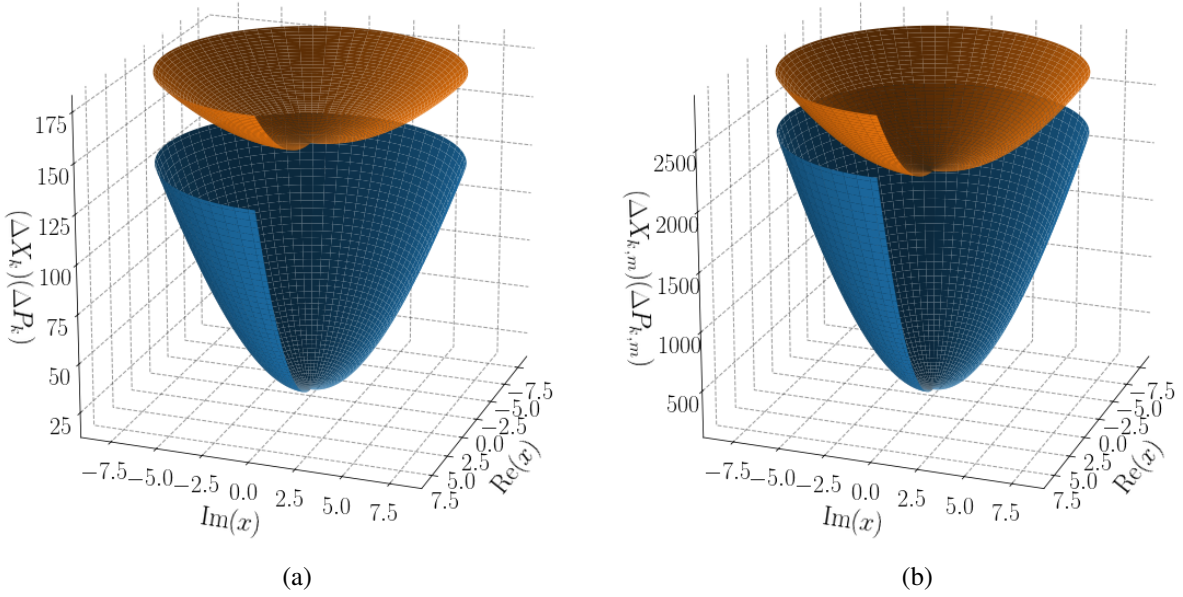
Figure 6: Uncertainty relations for the CS of the first-order SUSY partner of the trigonometric Pöschl-Teller potential with $\epsilon = -1/2$, for the quadratures of the annihilation and creation operators ℓ_k, $\ell_k^\dagger$ (a). The minimum values achieved are $125/8$ for $|w\rangle_0$ and $1145/8$ for $|w\rangle_1$. In (b) the corresponding multiphoton operators are used, and the minimum values obtained are $1275/8$ for $|w\rangle_0$ and $4095/2$ for $|w\rangle_1$. The potential parameter taken is $\nu = 2$.

potential, while it is linear in the quantum number n for the trigonometric Pöschl-Teller potential. The multiphoton algebra produces in a natural way the division of the spectrum of H_0 into m infinite energy ladders, consequently the corresponding Hilbert space is splitted into m orthogonal subspaces whose direct sum is the original Hilbert space. It is worth noting that the spectrum of the kth-order SUSY partners H_k have k extra single-step ladders, additional to the spectrum of the initial Hamiltonian H_0. The derived multiphoton coherent states for the harmonic oscillator potential reduce to the standard ones and to the "even" and "odd" states for the right choice of the mth power of the ladders operators. Finally, it would be interesting to evaluate the uncertainty relations for the physical position and momentum operators of our system, a task that will require extensive numerical calculations which are outside the scope of this paper.

Acknowledgments

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